Journal of Mathematics VOL.2 NO.1 March 2016
ISSN 2380-2634 (Print) ISSN 2380-2650 (Online) http://www.researchpub.org/journal/jm/jm.html

Thinning on the Cellular Complexes: Hexagonal and Quadratic

A. Trejo[1], *TES-OEM**

***Abstract*— This work is localized in two areas of the mathematics related with the processing of digital images, in particular with methods of complexes process or gridding for cells: digital topology and digital geometry. A proposed thinning algorithm in 2001 by Kovalevsky, to $2-$dimensional binary digital images modelled for cellular complexes is experimented on the hexagonal and quadratic cellular complexes. To the hexagonal and quadratic complex the Kovalevsky algorithm is developed and implemented as a mapping method of patrons inside of this work. The skeletons or complexes obtained in several experiments are analyzed with reference to some topological and geometrical properties and are compared with the Blum complex.**

***Keywords* —Consecutiveness, digital topology, Grid cell topology, Hexagonal and quadratic cellular complexes, Kovalevsky complex, pixel, thinning algorithm.**

I. Introduction

A Digital image is modeled as function whose definition domain is a discrete bounded subset of D, in $\Re^n$, and their values domain is a bounded subset of Z. The elements of the set D, are called pixels.

Inside of this work, we consider only digital images of dimension two. A digital image is the result of the application of a mapping called digitalizing. The necessity of the digitalizing arises of the process wish images with the help of computers and likewise can stored and treated. The digital processing of images IDP, was originally an area of the engineering, but this now deeply related with much areas of mathematics has in special relation with the topology and geometry given place to new research lines such as digital topology and digital geometry. Their goal is contribute to the formal analysis and to the understanding of the patrons present in images with help of the computer. For it, IDP, have all methods, as well as theoretical and technical to process digital images from their acquisition including their treatment with the objective of some improvement, and techniques to find objects of interests inside the images.

Here is studied a particular and interesting method of IDP, called *thinning*, which is important in analysis, classification and recognition of objects as are for example letters and figures, parts of technical draws, parts of animals, parts of plants or microorganisms, but also can be the “free space” of a mobile robot. The thinning is one of several methods to determine a subset called skeletal of each object of interest. Intuitively, the skeletal is a “ideally thin” version of the object, which preserve topological properties of the object as their *connectivity*, (see Appendix A) or their holes number (*genus*) but also preserve the geometrical properties as the length of arms. In 2001, Kovalevsky proposed a thinning algorithm [1], to *binary digital images* modelled for *cellular complexes*. It is worth mentioning that the proposal of the algorithm is very general and short, which not mentioned details of how to realize the central steps of the algorithm to certain specific cellular complex. This work has for goal develop and program the thinning algorithm of Kovalevsky to the quadratic and hexagonal cellular complexes, and of this way to do a comparing between the achieved skeletal for the algorithm and the Blum skeletal to a same object.

The programs realized in the margin of this work are implemented on the programming platform $DIAS$, (Digital Image Analysis System) which was developed by the group of the digital processing of images, under direction of Prof. K. Voss, in the Mathematics and Engineering Faculties of the Friedrich Schiller Universität of Jena, Germany. This programming pocket is free to be accessed in (http://pandora.inf.unijena.de/p/d/diasl.html).

II. Blum Skeletal

The definition of Blum skeletal was proposed and developed by Blum (also is called as transforming to the half-axis) in the year 1967 [2], which transforms a set R, (closed connected non-vanishing and bounded subset of $\Re^2$) on S, the skeletal of R. Being $\Re^2$, a metric space, we can to obtain of the S, elements of the following way:

With the Euclidean metric of $\Re^2$, if p, is a skeletal point of

[1]Tecnológico de Estudios Superiores del Oriente del Estado de México, Edo. de Méx. 56400. aalfredot@yahoo.com.mx

R, then p, is the center of a circumference of maximum radius that is included in R. This means that p, is the center of a included disc in R, and p, is not belongs to nothing disc more big (of different radius) included in R.

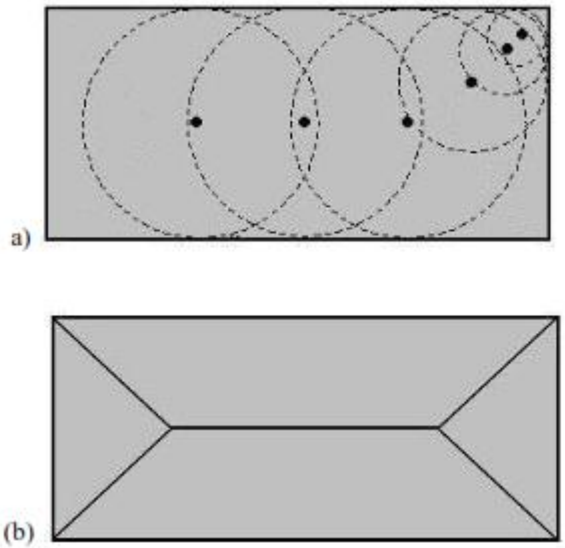


Fig. 1. In the figure a), are had some of the maximal circumferences included in the object. In the incise b), is showed the set of all centers of the circumferences as in a), said set is the Blum skeletal of the given rectangle.

It should be noted that to obtain the Blum skeletal to Euclidean object is not known some geometrical construction (with rule and measure) or some algorithm. Only is had the definition [[3], page 26], which no have a method to construct to the skeletal and is known that the skeletal is included in the set of maximal circumference centers that fit into the object. Likewise, in the experiments that was realized in this work, the Blum skeletal was constructed of approximated way, through the determining of said set of maximal circumferences centers in some manual experiments (with pencil and paper) in other with experiments of simulated manner as is showed in the figure 1.

One of proposes is argument through examples that the definition of Blum skeletal or half-axis, is not applicable to digitalized objects on digital plane.

In this work we take as digital plane the topological subspace Z^2, that as topological subspace of R^2, is a discrete topological space. The Euclidean plane is decomposed in squares, where each center of unitary square is identified through a bench, each square is called pixel which is known as the pixel plane ([4], page 21).

The elements of the digital plane (see the Appendix B) are related through some of the following relations of neighborhood:

Let $x=(x_1,y_1)$, and $y=(x_2,y_2)$, elements of Z^2, then:

1). y, is called $4-$neighbor of x, if y, belongs to the set

$$\{(x_1+1,y_1),(x_1-1,y_1),(x_1,y_1+1),(x_1,y_1-1)\}, \quad (1)$$

The set

$$N_4(x)=\{y\in Z^2 : y=x, \text{ or }, y, \text{ is } 4-\text{neighbor of } x\}, (2)$$

2). y, is called $8-$neighbor of x, if y, is $4-$neighbor of x, or y, belongs to the set

$$\{(x_1+1,y_1+1),(x_1-1,y_1-1),(x_1+1,y_1-1),(x_1-1,y_1+1)\}, \quad (3)$$

The set

$$N_8(x)=\{y\in Z^2 : y=x, \text{ or }, y, \text{ is } 8-\text{neighbor of } x\}, (4)$$

is called $8-$neighborhood.

On the digital plane are considered the following metrics:

i) To $x,y\in Z^2$,

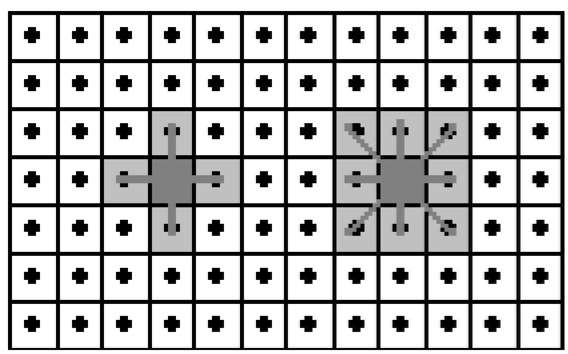
$$d_4(x,y)=|x_1-x_2|+|y_1-y_2|, \quad (5)$$

is called "Manhatan metric".

i) To $x,y\in Z^2$,

$$d_8(x,y)=\max\{|x_1-x_2|+|y_1-y_2|\}, \quad (6)$$

is called "Ajedrez metric".



a) 4 - **neighborhood** b) 8 - **neighborhood**

Fig. 2. Neighborhood relations.

With basis in the relations (5) and (6) we can define the following figures:

Let $r>0$, and $x\in Z^2$.

ISSN 2380-2634 (Print) ISSN 2380-2650 (Online) http://www.researchpub.org/journal/jm/jm.html

a) We define the closed disc with radius r, around x, as the set

$$D_{r,k}(x) = \{y \in Z^2 : d_k(x,y) \le r\}, \qquad (7)$$

to $k \in \{4,8\}$.

b) We define the circumference with radius r, around x, as the set

$$C_{r,k}(x) = \{y \in Z^2 : d_k(x,y) = r\}, \qquad (8)$$

to $k \in \{4,8\}$. See the figure 3.

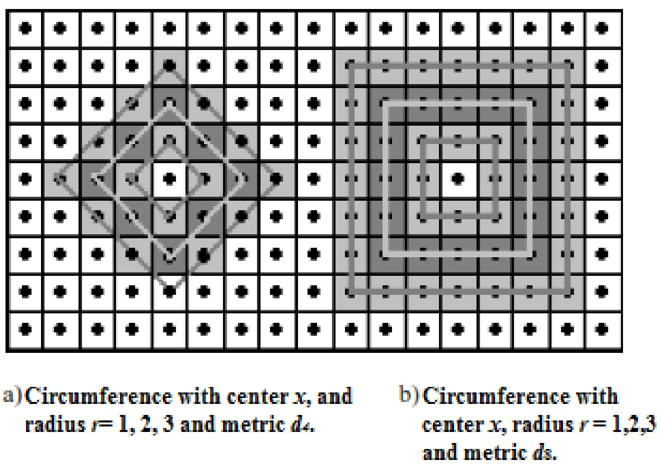


Fig. 3. The circumference.

The digitalization of a subset of the Euclidean plane, that also we can call object, is the capture of this through some mapping to the digital plane. The result is a subset of the digital plane called digital object. In [4], are have established two digitalization mappings developed by C. F. Gauss and C. Jordan.

i) Let M, be subset of the Euclidean plane. The Gauss digitalization $G(M)$, is the union of the squares (pixels) whose center is included in M.

ii) Let M, be a subset of the Euclidean plane. Let $J_h^-(M)$, the union of all squares (pixels) that are completely included in M, and $J_h^+(M)$, the union of all squares which intersects to M. $J_h^-(M)$, is called the Jordan inner digitalizing of M, and $J_h^+(M)$, is called the Jordan exterior digitalizing of M.

See the figure 4.

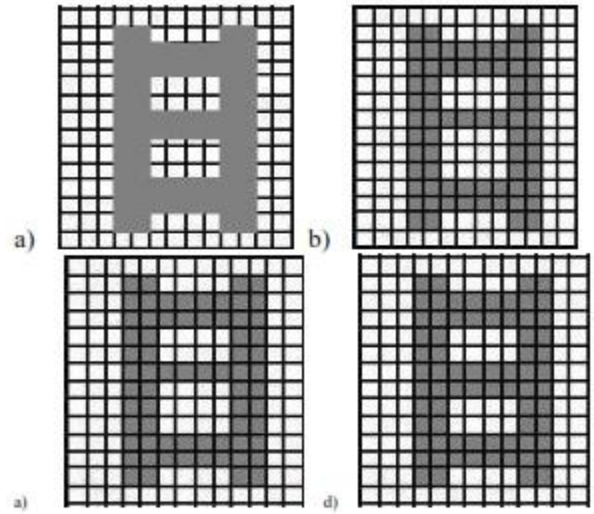


Fig. 4. In the figure a), is showed the Euclidean object y, in the figure b), is has the result of the Gauss digitalizing, likewise in c) is has the Jordan inner digitalizing and in d), the Jordan exterior digitalizing.

Taking in consideration some of the digitalizing processes we can apply the definition of Blum skeletal to the digital plane, since we had defined the concept of circumference in the digital plane. We observe the figure 5.

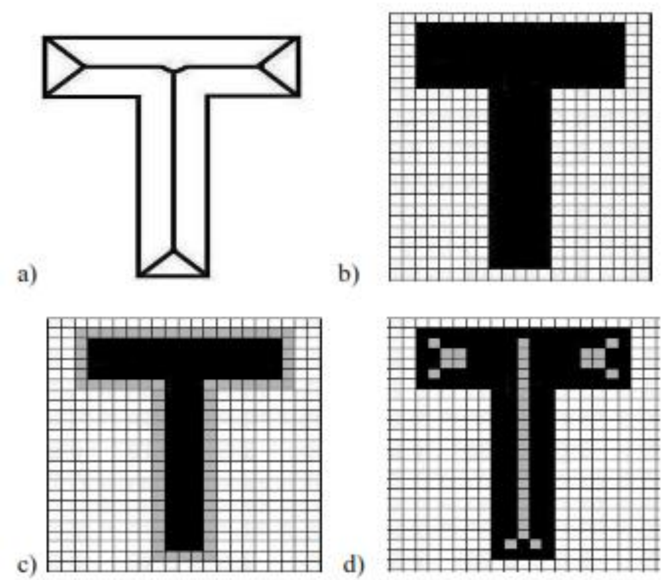


Fig. 5. In a) is showed the Blum skeletal of the object of the Euclidean space. In b), is has the object digitalizing through the Gauss digitalizing. In c), is identified the contour of the digitalized object, and d), is has the Blum skeletal to the digitalized object.

As we can observe the definition of Blum skeletal is not adequate to identify digital objects, since the skeletal we must permit determine topological and geometrical properties of the object as the connectivity and shape, and in the example of the figure 5, these properties are lost.

III. Modelling of the Digital Image as Cellular Complex

The abstract cellular complexes are known in the combinatory topology and in the polyhedral geometry. The use of this model to describe the topology and geometry of digital objects was treated for first time by Kovalevsky in the eighty years [1].

To understand the structure of a cellular complex we analyzes the figure 6. These consist of elements of different dimensions over the Euclidean plane: faces (out cantos), cantos (out final points) and vertices. Furthermore, we have that a canto a_1, bounds or borders to the faces c_1, and c_2, having that also the vertices υ_1, and υ_2, bound or border to the canto a_1, and the faces c_1, and c_2. The formed set for these all elements is known as cellular complex and their elements are called cells. It´s important to empathize that an element of dimension 2, is the inner of a convex polygon, an element of dimension 1, is a line segment which is considered out their two final points. An element of dimension 0, is an only point.

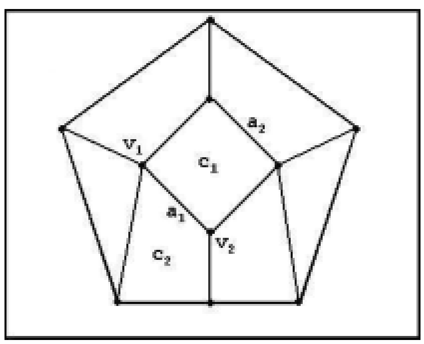
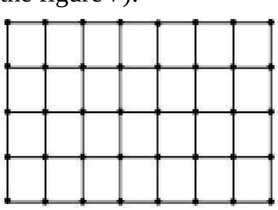
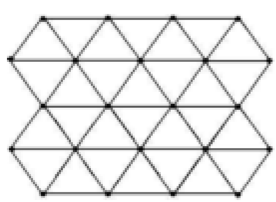


Fig. 6. Cellular complex.

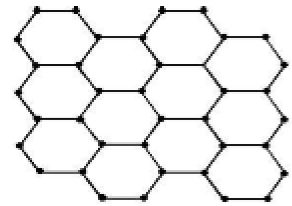
To decompose the Euclidean plane in regular convex polygons (all their angles and sides equals), results that the unique possible regular convex polygon are: the square, triangle and hexagons ([3], page 45-46). Likewise are had three different complexes to consider: quadratic cellular complex, triangular cellular complex and hexagonal cellular complex (see the figure 7).



a) Quadratic Cellular Complex



b) Triangular Cellular Complex



c) Hexagonal Cellular Complex

Fig. 7. Cellular complexes.

All experiments, from the digitalizing of objects until the study of the Kovalevsky skeletal, will be realized exclusively in the quadratic cellular complex, and the hexagonal cellular complex. The triangular cellular complex is treated in [8].

To digitalize Euclidean objects over mentioned cellular complex we generate cellular sub-complexes applying the following method:

In the digital plane is considered the pixel as unitary square whose center belongs to Z^2. Generalizing this idea to the quadratic or hexagonal cellular complexes is had that: is considered as pixel to the cells of major dimension to use the Gauss or Jordan digitalizing mappings where each cell is identified with their center. One time that is had information of the cells of dimension 2, that conform to the object, is defined which cells of minor dimension are aggregated to the digital object. For each 2-dimensional cell is conformed the digital object, this will be accompanied of all their cantos and vertices to achieve the preservation of the connectivity of the original object.

With the object of can visualize each one of the methods before mentioned, in the following Table 1, is explained as is managed each one of the complexes inside of PC.

By agreeing to the established in the Table 1, we present the visualizing of a bound region of the quadratic and hexagonal complex (see the figure 8).

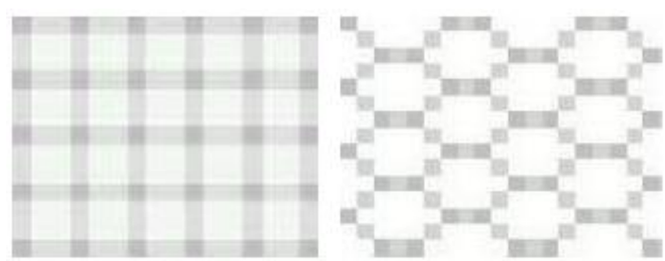


Fig. 8. More cellular complexes in combined geometry pixels to quadratic and hexagonal complexes.

In the figure 9, are showed the results that were obtained as results of the algorithm programming of digitalizing on the cellular complex. Showing firstly, the Euclidean object to digitalize and after are showed the 2-dimensional cells belonging to the digitalizing of the object, obtaining finally all cells of minor dimension that structure to the digitalization.

Journal of Mathematics VOL.2 NO.1 March 2016
ISSN 2380-2634 (Print) ISSN 2380-2650 (Online) http://www.researchpub.org/journal/jm/jm.html

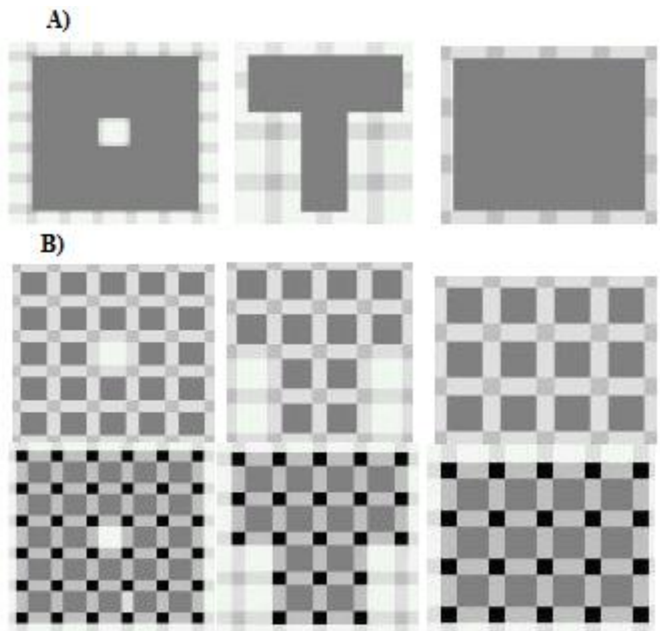


Fig. 9. Digitalizing on quadratic complex. A). Series of Euclidean Objects. B). Series of digital objects (Gauss and Exterior Jordan digitalizing respectively) on the hexagonal cellular complex

From the figure 9, and 10, we have that when is digitalized a same object in each cellular complex, the result is geometrically different, since the original shape of the object is not conserved in the hexagonal complex and in the quadratic complex yes it is. This fact is relevant, since to the implementation of the thinning algorithm on the cellular complex is determinant the digitalizing with which is initiated the algorithm, this to obtain a good approximation of the Blum skeletal.

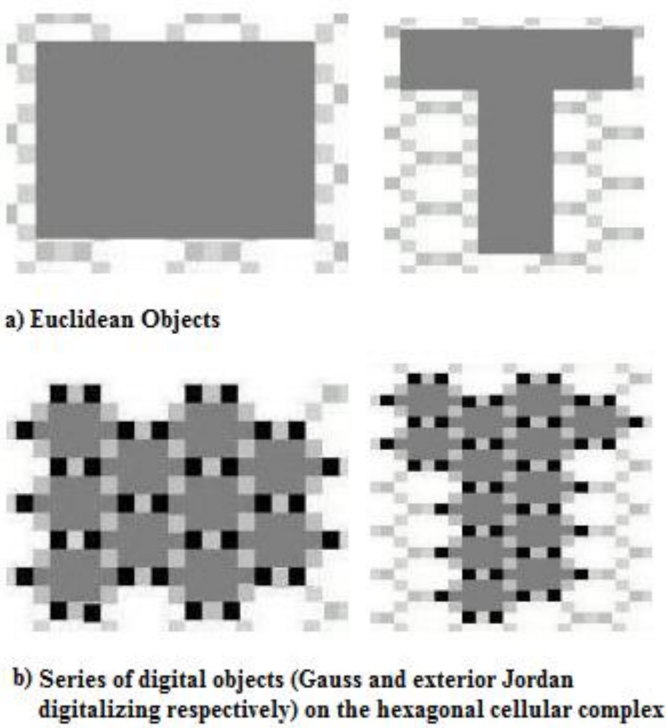


Fig. 10. Digitalization over hexagonal complex.

TABLE I
VISUALIZATION OF THE QUADRATIC AND HEXAGONAL CELLULAR COMPLEXES IN THE PC

Element of the quadratic complex	Euclidean plane representation	Monitor Representation
0-dimensional cell	Point	Pixel
1-dimensional cell	Edge or Canto	Set of Two Pixels
2-dimensional cell	Square	Set of Four Pixels
Hexagonal complex element	Euclidean Plane Representation	Monitor Representation
1-dimensional cell	Edge or Canto	Pixel
2-dimensional cell	Hexagon	Set of Four Pixels

IV. THINNING IN CELLULAR COMPLEXES AND THEIR IMPLEMENTATION IN SOME RESULTS

Historically, Listing handled in 1862 the term of linear skeletal to describe to the resultant subset of continuous deformations of a connect subset of a Euclidean space no change in the connectivity of the original set until to obtain a set of points and lines ([4], page 215). Much algorithms of image analysis are based in this deforming process called thinning.

In the literature the term “thinning” has not an unique interpretation, although describe reduction operations that preserve the connectivity to be applied on digital objects involving iterations to transform border elements in fund elements.

A thinning algorithm must to satisfy the following conditions as has been established in several publications, see for example [5]:

i). The resulting object of the original object must be thin.
ii). The subset must be approximated to the middle axis.
iii). The final points must be preserved. The final points are those points that are extreme of lines or digital curves. If they is not preserved in a digitalizing process, the lines or digital curves degenerate in points.
iv). The connectivity of the object and the fund must be preserved. The pixels of the bordering or open bordering of a digital object whose eliminating preserve the connectivity of

Journal of Mathematics VOL.2 NO.1 March 2016
ISSN 2380-2634 (Print) ISSN 2380-2650 (Online) http://www.researchpub.org/journal/jm/jm.html

the object and their fund, is known with the name of simple pixels. There is different ways to localize these pixels in the digital object: templates and Boolean expressions.

v). The algorithm must be robust versus noise.

We can talk of bordering when the contour of the digitalized object over cellular complex is structured of cells of 0-and 1-dimension (points and segments). This set is denoted as Fr, and is talked of open bordering when the contour of the digitalized object over some cellular complex is structured of 0- and 2-dimension cells (points and squares or hexagons). This set is denoted as Of.

We mention that to the elimination of simple pixels of the bordering or of the open bordering, will be used templates. A template is a pixel arrangement around of a pixel of interest (neighborhood of pixels) that permit to realize local operations through the comparing of present values in the templates with the values of the digital object. These operations search different goals between which is the preservation of the connectivity of the digital object. We mention that to detect a final point and their preservation in a digital object, also is implemented templates (see the figure 11).

The established templates in ([3], page68-73) and applied in the algorithm were established of the following way: each element of the hexagonal complex, in the PC is a pixel or well, a pixel set. From the TABLE I, we have that the visual representation of a point and one edge in the PC, to this complex are of a pixel and to the hexagon is a set of 8 pixels in a particular arrangement, all assigned with a value as is case: 1-object or 0-fund (see the appendix B).

To analyze the incident elements with a point or a hexagon canto or edge is considered arrange of 8 pixels, around of the pixel of interest (see the Appendix B) that represents a point or an edge that furthermore, in arrangement is situated in the center.

1	1	1
0	1	1
0	0	0

0	0	0
0	1	1
1	1	1

1	0	0
1	1	0
1	0	0

1	1	1
1	1	0
0	0	0

Fig. 11. Examples of templates to delete elements of the bordering.

To the set that represents the hexagon we have an arrangement composes of 23 pixels, 11 that represent and 12 that represent the points and incident edges of cantos with him (see the figure 12).

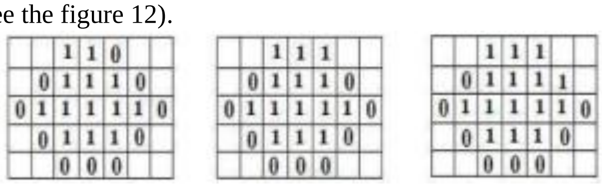


Fig. 12. Examples of templates to hexagons of the boundering.

The corresponding templates to the quadratic complex were obtained through analogous process to the before (see [7], page 55).

In base of the explained information we have that the steps that were considered to the development of the thinning algorithm to the quadratic and hexagonal complex are the following (see the algorithm ([7], page 66-74)):

STEPS

1. The object of our interest of the Euclidean plane is digitalized to a subset (denote it as T) of the cellular complex C, through the digitalized established methods.
2. A new sub-complex is generated $T_1 = T \cup Fr(T)$, to guarantee that the enter sub-complex to the algorithm is closed.
3. The elements of $Fr(T_1) \cup T_1$, that comply with some of the established configurations of templates, are searched and benched.
4. The benched elements in the step 3, are deleted and counted, generating a sub-complex T_2.
5. The elements of $Of(T_2) \cup T_2$, that comply with some of the template configurations are searched and benched.
6. The benched elements in the step 5, are counted and deleted, generating a sub-complex S.
7. The number of counted elements in the steps 4, and 6, is summed. In this moment finalize one iteration.
8. If the sum of the before step is zero, finalize the iterations and is reported the subset S, that represents to the skeletal of the sub-complex T. In the contrary case, is actualized $T_1 = S$, and begins a new iteration, beginning in 3, until comply the condition of the end of the algorithm.

To continuation are showed four figure blocks (see the figures 13-16) in which only in the two first is presented the Blum skeletal or middle axis, to be compared with the result of the application of the algorithm. In the following only is presented the result of the application of the algorithm.

The order of the figure in each block is the following: the Blum skeletal, digitalization of the Euclidean object, digitalized skeletal over hexagonal complex and digitalization of the skeletal over quadratic complex (see the figures 13-16).

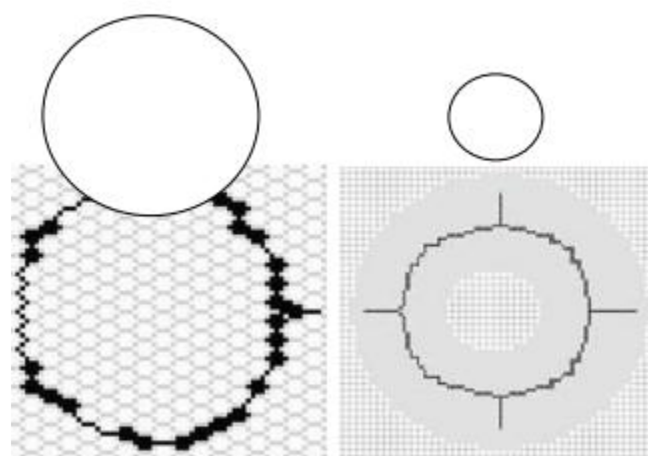


Fig. 13. Digitalized skeletal of “O” letter over hexagonal complex and quadratic complex.

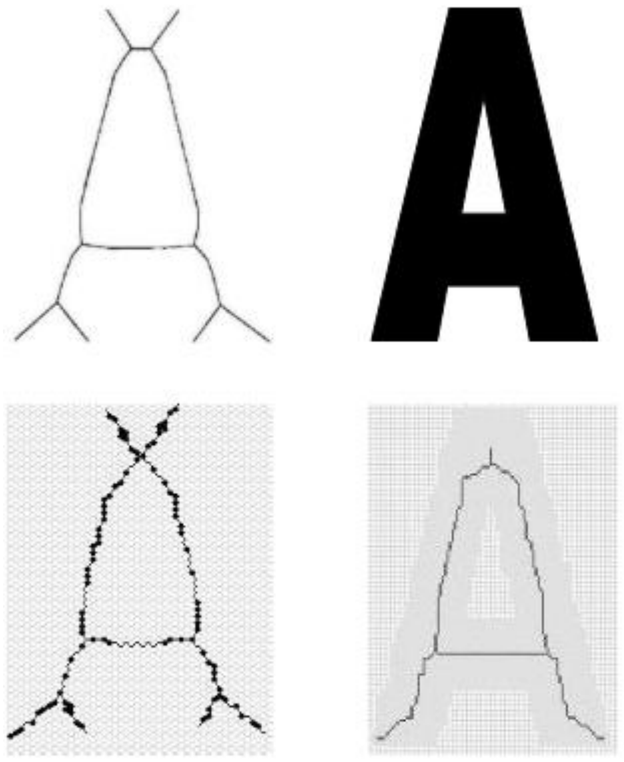


Fig. 14. Digitalized skeletal of "A" letter over hexagonal complex and quadratic complex.

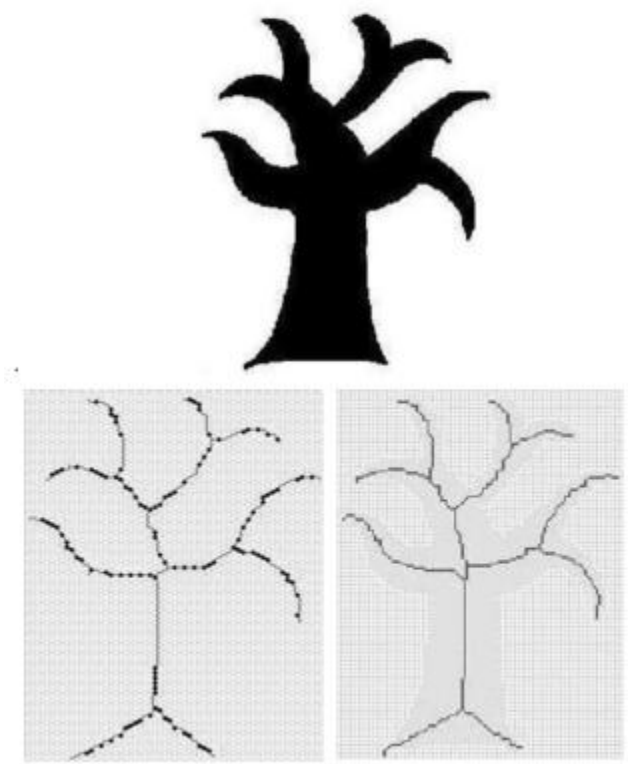


Fig. 15. Digitalized skeletal of a ramifications series over hexagonal complex and quadratic complex.

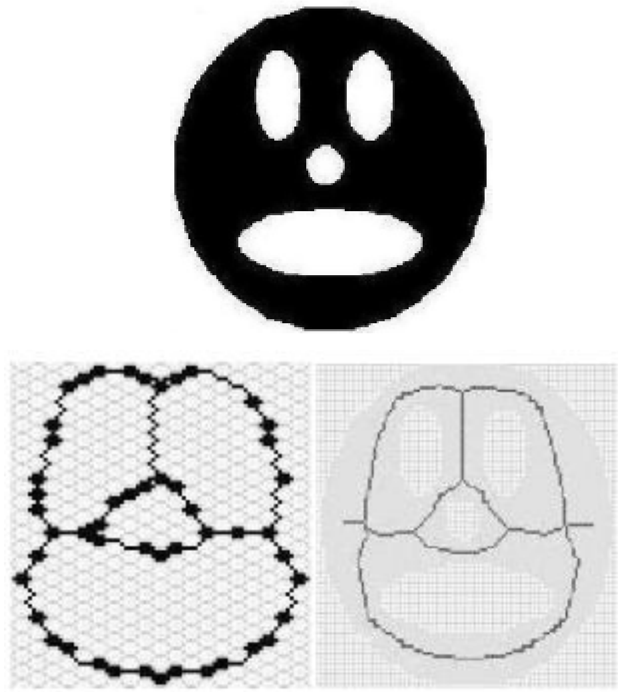


Fig. 16. Digitalized skeletal of a circles series over hexagonal complex and quadratic complex.

From the figures 13-16, is observed the following:

i. In the digitalized skeletal of the Euclidean object, in any of the cellular complexes are originated branches that in the original object no appear (see the figure 13). This is due to the noise generated for the digitalizing process.

ii. The digitalized skeletal over the quadratic complex no achieves to conserve the total number of "corners" of the original object. To the hexagonal complex this number yes is achieves to conserve in the digitalized skeletal (see the figure 14).

iii. The points of bifurcations are conserved in the digitalized skeletal in the quadratic complex and in the hexagonal complex (see the figure 15).

iv. The number of holes of the original object is totally conserved in the digitalized skeletal of both cellular complexes (see the figure 16).

v. In all the presented examples the skeletal conserve the connectivity of the object, that is to say, the skeletal no stay divided in fragments.

From the observations we have that the best results presented were obtained in the hexagonal complex, since is generated minor noise, and wit it is got a good approximation to the half axis (Blum skeletal) as can see it in the figure 13 and figure 14.

V. Conclusions

According to the results that were obtained in the before section is concluded the following: The methos of digitalization presented in the section IV, applied to Euclidean objects ti the cellular complexes are very sensitive under perturbations very little of the original object such and as is showed in the figures 13, 14, and 16.

To digitalized objects to a cellular complex only is can report that the connectivity is a property that is guaranteed to be preserved under the established algorithm, that is to say, this algorithm to each connected sub-complex produces a connected skeletal. Furthermore, this algorithm no generates holes in the object and no connects parts of the object that originally no were connected. The experiments reported showed that with other properties is necessary to have much careful in the interpretation of the number of final elements, likewise also the number of bifurcation elements.

Also we can to say, that the Blum skeletal of each one of our computational proofs, when are compared with the generated skeletal as consequence of the application of the achieved program, have a big similitude, which is major to the hexagonal complex (see the figures 13, and 14). However here we have present problems caused for digitalizing effects.

Finally, the following observations are not tried in this work, but yes in ([3], chapter 5), where is relevant to mention:

a). The discretizing resolution plays a primordial role to that geometrical and topological properties of the Euclidean object are reflected in the digitalized object.

b). Negative effects of the digitalizing are observed when the Euclidean object is translated or rotated, since a translation or rotation generates that the contour of the digitalized object varies.

Appendix A

Some Thinning Algorithm Theorems.

In a general context, thinning means reducing the training data set $\{X,Y\}$, to a smaller subset $\{X',Y'\}$. The classifier then only uses $\{X',Y'\}$. This results in reduced memory requirements and query times. There is an important property of thinning data sets $\{X',Y'\}$,

Def. A. 1. A set $\{X',Y'\}\subseteq\{X,Y\}$, is called consistent subset of $\{X,Y\}$, if the $1-NN$ classifier for $\{X',Y'\}$, correctly classifies all members of the original set $\{X,Y\}$.

We extend this definition to a $k-NN$ classifier.

Def. A. 2. A vector $x\in X$, is called $k-$consistent with respect to $\{X,Y\}$, if the unanimous $k-NN$ classifier for $\{X,Y\}$, classifies it correctly. Otherwise it is called $k-$inconsistent with respect to $\{X,Y\}$. A set $\{X,Y\}$, is called $k-$consistent set if it has no elements which are $k-$ inconsistent with respect $\{X,Y\}$. A subset $\{X',Y'\}\subseteq\{X,Y\}$, is called $k-$ consistent subset of $\{X,Y\}$, if all members of $\{X,Y\}$, are $k-$consistent with respect to $\{X',Y'\}$.

The terms consistent subset and 1-consistent are subset equivalent. As for the $1-NN$ case, the property $k-$consistent subset guarantees perfect recognition of the $k-NN$ classifier for $\{X',Y'\}$, applied to the whole training set $\{X,Y\}$.

Theorem A. 1. A vector which is $k-$consistent with respect to a set is also $k'-$ consistent with respect to the same set for all $k'\leq k$.

Proof. The k, nearest neighbors of a labeled vector (x,c), which is $k-$consistent with respect to $\{X,Y\}$, are all in class c. Thus, also its k', nearest neighbors are in class c. Thus (x,c), is $k'-$consistent with respect to $\{X,Y\}$. ◆

The proof of the before theorem gives the algorithm:

Input: $\{X,Y\}$		
Initialize R, with one Random element of $\{X,Y\}$		
FOR EACH $(x,c)\in\{X,Y]/R$		
	IF	x, is $k'-$inconsistent with respect to R
	THEN	Set $R=R\cup(x,c)$

Theorem A. 2. A $k-$consistent subset of a set is a $k-$consistent set.

Proof. Given a $k-$ consistent subset $\{X',Y'\}$, of $\{X,Y\}$, all elements $\{X,Y\}$, are $k-$consistent with respect to $\{X',Y'\}$. As $\{X',Y'\}\subseteq\{X,Y\}$, then all elements of $\{X',Y'\}$, are $k-$consistent with respect to $\{X',Y'\}$. Thus $\{X',Y'\}$, is a $k-$consistent set. ◆

Theorem A. 3. A $k-$consistent subset of a set is a $k'-$consistent subset of the same set for all $k'\leq k$.

Proof. Let $\{X',Y'\}$, be a $k-$consistent subset of $\{X,Y\}$. All labeled vectors in $\{X,Y\}$, are by definition $k-$consistent with respect to $\{X',Y'\}$, and thus $k'-$consistent with respect to $\{X',Y'\}$, (theorem A. 1). Thus $\{X',Y'\}$, is a $k'-$consistent subset of $\{X,Y\}$. ◆

Appendix B

$(8,4)-$ **Digital Picture and Delete Rules to bordering elements.**

We use the fundamental concepts in digital Topology.

Let p, be a pint in the 2-dimensional digital space Z^2. Let us denote $N_m(p)$, the set of points that are $m-$ adjacent

($m = 4,8$) see the figures 11 (where the center is occupied by p). Note that, throughout of this paper some figures are depicted on the square complex that is dual to Z^2.

The equivalence classes relative to the $m-$ connectivity relation (that is to say, the transitive closure of the reflexive and symmetric $m-$ adjacency relations) are the $m-$ components of a set of points $X \subseteq Z^2$.

A $(8,4)$, digital picture P, is a quadruple $(Z^2, 8, 4, B)$. Each element of Z^2, is said to be a point of P. Each point in $B \subseteq Z^2$, is called black point or 1-object (binary code). Each point in Z^2 / B, is said to be a white point or 0-fund (binary code). A black component is an 8-component of B, while a while component is a 4-component of Z^2 / B.

A black point is called a border point in a $(8,4)$, picture if it is $4-$ adjacent to at least one white point. A black point in a picture is said to be an interior point if it is not a border point.

A reduction (on a 2D picture) is topology-preserving if each black component (as a set of points) in the original picture contains exactly one black component of the produced picture, and each white component in the output picture contains exactly one white component of the input picture.

A black point or 1-object is simple in a picture if only if its deletion is a topology-preserving reduction. We mention now the following characterization of simple points:

Theorem B. 1. A black point p, is simple in a picture $(Z^2, 8, 4, B)$, if and only if all of the following conditions hold:

i). $N_8(p) \setminus \{p\}$, contains exactly one black component.

ii). p, is a border point.

Proof. [10, 11]. ◆

Recall that a deletion rule is equivalent if it yields a pair of equivalent parallel and (order-independent) sequential reductions. Also are given the following sufficient conditions for equivalent deletion rules:

Theorem B. 2. Let R, a deletion rule. Let $(Z^2, 8, 4, B)$, be an arbitrary picture, and let $q \in B$, be any point that is deleted from that picture by R. Deletion rule R, is equivalent if the following two conditions hold for any $p \in B \setminus \{q\}$:

i). If p, can be deleted from picture $(Z^2, 8, 4, B)$, by R, then p, can be deleted from picture $(Z^2, 8, 4, B \setminus \{q\})$, by R.

ii). If p, cannot be deleted from picture $(Z^2, 8, 4, B)$, by R, then p, cannot be deleted from picture $(Z^2, 8, 4, B \setminus \{q\})$, by R.

Reductions associated with parallel thinning phases may delete a set of black points (1-objects) and not just a single simple point. Hence we need to consider what is meant by topology preservation when a number of black points are deleted simultaneously. Many author proposed sufficient conditions for reduction to preserve topology []. A contition is given by the following result:

Theorem B. 3. A (parallel) reduction with deletion rule R, is topology-preserving if the following conditions are given:

a). R, is equivalent.

b). R, deletes only simple points.

Proof. [9-12]. ◆

VI. Acknowledgments

Acknowledgments

I am grateful with the Editor-in-Chief of JM, for his invitation to contribute in this prestigious journal.

References

[1] V. A. Kovalevsky, Algorithms and data structure for computer topology, *Digital and Image Geometry*, (G. Bertrad, A. Imiya, R. Klette, Eds), Lecture Notes in Computer Science, Vol. 2243, pp37-38, Springer Verlag, Berlin Heidelberg, 2001.

[2] H. Blum, A transformation for extracting new descriptors of shape, W. Wathen-Dunn (ed), *Model for the perception of speech and visual form*, MIT Press, Cambridge, MA, USA., pp362-380, 1967.

[3] A. Trejo, Geometría del adelgazamiento sobre el complejo celular cuadrático y sobre el complejo celular hexagonal, Master Thesis, Automatic Control Department, CINVESTAV-IPN, Mexico D. F., December 2007.

[4] R. Klette, A. Rosenfeld, Digital Geometry: *Geometric Methods for Digital Picture Analysis.* Morgan Kaufmann Publishers, Elsevier, CA, USA., 2004.

[5] G. Klette, *Skeletons in Digital Image Processing*, Thechnical Report CITR-TR-112, Computer Science Department of the University of Auckland, July, 2002.

[6] Manual de Usuario DIAS (version-1998), FSU Jena, Tower-Soft, Berlin. (http://pandora.inf.unijena.de/p/d/diasl.html).

[7] M. R. Zempoalteca, "Adelgazamiento en imágenes digitales de dimension 2, modeladas por complejos celulares cuadráticos," Master Thesis, Automatic Control Department, CINVESTAV-IPN, Mexico D. F. 2004.

[8] P. S. Morales-Chávez, "Topología del adelgazamiento sobre el complejo celular cuadrático y sobre el complejo celular triangular," Master thesis, Automatic Control Department, CINVESTAV-IPN, Mexico D. F. 2005.

[9] L. S. Davis, *Parallel Image Analysis: Theory and Applications*, Volume 1, World Scientific Series in Machine Perception and Artificial Intelligence Vol. 19, (L. S. Davis, K. Inoue, M. Nivat, A. Rosenfeld, P. S. P. Wang), MA, USA, 1996.

[10] P. Kardos, K. Palágyi, On Topology Preservation in Triangular, Square and Hexagonal Grids, Proc. 8th Int. Symposium on Image and Signal Processing and Analysis, ISPA, pp. 782-787 (2013).

[11] T. Y. Kong, "On Topology Preservation in 2-d and 3-d Thinning," International Journal of Pattern Recognition and Artificial Intelligence 9, pp.813-844 (1995).

[12] C. Ronse, "Minimal Test Patterns for Connectivity Preservation in Parallel Thinning Algorithms for Binary Digital Images," Discrete Applied Mathematics 21, pp.67-79 (1988).